Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
MTH101A, Mid-Term Examination Grading Scheme
February 20, 2013

Marks: 60
Time: 2 Hours

***Answer all questions.* All the parts of each question must be answered in continuation, otherwise they will not be graded.**

1. (a) Using Sandwich theorem, determine the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, where

$$x_n = \frac{n^3}{2n^4+3n+1} + \frac{n^3}{2n^4+3n+2} + \ldots + \frac{n^3}{2n^4+4n}$$ [5]

Solution: $n \times \frac{n^3}{2n^4+4n} \le x_n \le n \times \frac{n^3}{2n^4+3n+1}$ …………………(2 marks)

$\Rightarrow \lim_{n\to\infty} x_n = \frac{1}{2}$. ……………………………………………… (3 marks)

(b) Prove that the sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_{n+1} = \sqrt{5 s_n}$, $s_1 = 1$ is convergent. [5]

Solution: $\{s_n\}$ is increasing, since $s_{n+1} > s_n$. ………………………………………………(2 marks)

$\{s_n\}$ is bounded above by 5 since $s_1 < 5$ and $s_n < 5 \Rightarrow s_{n+1} < 5$. ……………..(3 marks)

Thus, $\{s_n\}$ is convergent.

2. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0-\delta, x_0+\delta)$. [5]

Solution: Since $f(x_0) - c > 0$, choose $\varepsilon > 0$ such that $0 < \varepsilon < f(x_0) - c$.

Since, f is continuous at x_0, for this choice of ε, there exists a $\delta > 0$, such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. ………………………………………………………… (2 marks)

Hence, for all $x \in (x_0-\delta, x_0+\delta)$, $f(x) > f(x_0) - \varepsilon > c$ …………………………(3 marks)

Alternatively, let $\delta > 0$ does not exist such that $f(x) < 0$ in interval $(c-\delta,\, c+\delta)$. Then, in every interval $\left(c - \frac{1}{n},\, c + \frac{1}{n}\right)$, $n > N$, there exists an $x_n \in (a,\, b)$, such that $f(x_n) \ge 0 \Rightarrow \lim_{n\to\infty} f(x_n) \ge 0 \Rightarrow f(c) \ge 0$, a contradiction.

(b) Using the definition of differentiability, determine the non-negative real numbers α for which the function $g(x)=x^{\alpha}\sin^2\frac{1}{x}$ *for* $x\neq 0,\ g(0)=0$ is differentiable at $x=0$. Give complete arguments justifying your answer. [5]

Solution: $\varphi(x)=\dfrac{g(x)-g(0)}{x}=x^{\alpha-1}\sin^2\dfrac{1}{x}$

$\left|x^{\alpha-1}\sin^2\dfrac{1}{x}\right|\leq|x|^{\alpha-1}\to 0\ as\ x\to 0,\ if\ \alpha>1\Rightarrow g(x)$ is differentiable at $x=0$ for $\alpha>1$. [1 marks]

For $\alpha=1$, $\lim\limits_{x\to 0}\varphi(x)=\lim\limits_{x\to 0}\sin^2\dfrac{1}{x}$ does not exist, since for $x_n=\dfrac{1}{n\pi}$, $\lim\limits_{n\to\infty}\sin^2\dfrac{1}{x_n}=0$ and for $x_n=\dfrac{2}{(2n+1)\pi}$, $\lim\limits_{n\to\infty}\sin^2\dfrac{1}{x_n}=1$. Therefore, $g(x)$ is not differentiable at $x=0$ for $\alpha=1$. [2 marks]

For $0<\alpha<1$, $\lim\limits_{x\to 0}\varphi(x)=\lim\limits_{x\to 0}x^{\alpha-1}\sin^2\dfrac{1}{x}$ does not exist since for $x_n=\dfrac{1}{n\pi}$, $\lim\limits_{n\to\infty}x_n^{\alpha-1}\sin^2\dfrac{1}{x_n}=0$ and for $x_n\neq\dfrac{1}{n\pi}$, $\lim\limits_{n\to\infty}x_n^{\alpha-1}\sin^2\dfrac{1}{x_n}=\infty$.

Therefore, $g(x)$ is not differentiable at $x=0$ for $0<\alpha<1$ [2 marks]

3. (a) Use Mean Value Theorem to determine $\lim\limits_{x\to\infty}\left[(x+2)\tan^{-1}(x+2)-x\ \tan^{-1}x\right]$. [5]

Solution. Let $f(x)=x\tan^{-1}x$. Then, f satisfies the conditions of MVT in $[x,\ x+2]$. (1 mark)

$\Rightarrow$ for some $c\in(x,\ x+2)$, $(x+2)\tan^{-1}(x+2)-x\tan^{-1}x=2\left[\tan^{-1}c+\dfrac{c}{1+c^2}\right]$.......(2 marks)

$\to 2\times\dfrac{\pi}{2}=\pi$ as $x\to\infty$ ....(2 marks)

(b) Prove that $x<\tan x$ for $x\in\left(0,\ \dfrac{\pi}{2}\right)$. [5]

Solution. Let $g(x)=\tan x-x$ ....(1 mark)

Then, $g'(x)=1+\tan^2 x-1>0\Rightarrow g(x)\uparrow\ in\ \left[0,\ \dfrac{\pi}{2}\right]$ ....(2 marks)

$\Rightarrow\ g(x)>g(0)=0\ \Rightarrow\tan x>x$ ....(2 marks)

4. (a) Let $a_n \geq 0$. Show that both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{a_n + 1}$ converge or diverge together. [5]

Solution: (i) $Since, 0 \leq \frac{a_n}{1+a_n} \leq a_n,$

$$\sum_{n=1}^{\infty} a_n \ convergent \Rightarrow \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} (by \ comaprison \, test) \text{ is convergent} \quad \text{.........} \quad \text{(2 marks)}$$

(ii) $Since, 1 \leq 1 + a_n \leq 2 \ eventually, \ 0 \leq \frac{1}{2} a_n \leq \frac{a_n}{1+a_n}$. Therefore,

$$\sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \ convergent \Rightarrow \sum_{n=1}^{\infty} a_n \ \ convergent \ \ (by \ comaprison \, test) \ \text{..............(3 marks)}$$

(b) Let $f(x) = \frac{2x^2 - 1}{x^2 - 1}$. Find (i) asymptotes of f (ii) Locate the intervals where f is increasing or decreasing (iii) Locate the points of local maxima and minima of f (iv) Locate the intervals where f is convex or concave (v) Sketch the graph of f. [5]

Solution: (i) Asympltotes at $y = 2, x = 1, x = -1$ .... (1 mark)

(ii) increasing on $(-\infty, -1) \cup (-1, 0)$, decreasing on $(0,1) \cup (1, \infty)$ ... (1 mark)

(iii) 0 is local maxima .... (1 mark)

(iv) convex in $(-\infty, 1) \cup (1, \infty)$ concave in $(-1, 1)$ .... (1 mark)

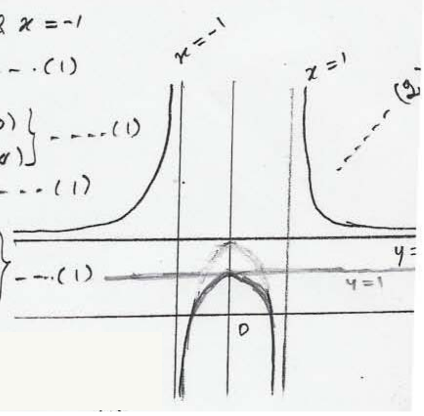
(v) correct sketch .... (1 mark)

Note that $f(x) = 2 + \frac{1}{x^2-1}$. Hence $y = 2$, $x = 1$ & $x = -1$ are asymptotes ---(1)

$f'(x) = \frac{-2x}{(x^2-1)^2} \Rightarrow$ f is increasing on $(-\infty,-1) \cup (-1,0)$, f is decreasing on $(0,1) \cup (1,\infty)$ ---(1)

& 0 is a local maximum ---(1)

$f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3} \Rightarrow$ f is convex on $(-\infty,1) \cup (1,\infty)$, f is concave on $(-1,1)$ ---(1)



5. (a) State Generalized Mean Value Theorem precisely. Using this theorem and the inequality $\sin x > x - \frac{x^3}{3!}$ for $x > 0$, prove that $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x > 0$. [5]

Solution: Correct Statement: ....(1 mark)

Let $f(x) = \sin x - x + \frac{x^3}{3!}$, $g(x) = \frac{x^5}{5!}$

GMT $\Rightarrow \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{\cos c - 1 + (c^2/2!)}{c^4/4!}$ .... (*).......................(2 marks)

Using GMT again, $\frac{\cos x - 1 + (x^2/2!)}{x^4/4!} = \frac{-\sin d + d}{d^3/3!} < 1$, (by the given inequality)........(2 marks)

$\Rightarrow$ Expression in (*) < 1. Alternatively, use Taylor's Theorem.

(b) Suppose a function $h(x)$ is three times differentiable function on $[-1, 1]$, $h(-1) = 0$, $h(1) = 1$ *and* $h'(0) = 0$. Use Taylor's theorem to prove that $h'''(c) \geq 3$ for some $c \in (-1, 1)$. [5]

Solutions: By Taylor's theorem,

$h(1) = h(0) + h'(0) + \frac{h''(0)}{2!} + \frac{h'''(c_1)}{3!}$ for some $c_1 \in (0, 1)$ (1 mark)

$h(-1) = h(0) - h'(0) + \frac{h''(0)}{2!} - \frac{h'''(c_2)}{3!}$ for some $c_2 \in (-1, 0)$ (1 mark)

$\Rightarrow \frac{h'''(c_1) + h'''(c_2)}{6} = 1$ (2 marks)

$\Rightarrow$ either $h'''(c_1)$ or $h'''(c_2) \geq 3$. (1 mark)

6. (a) Show that there is no continuous function $f(x)$ such that $\int_0^1 e^{n^2 x} f(x)\, dx = \frac{e^{n^2}}{n}$ for all n. [5]

Solution: Suppose such a function exists. Let $\sup_{x \in [0,1]} f(x) = M$. Then,

$$\frac{e^{n^2}}{n} = \left| \int_0^1 f(x) e^{n^2 x} dx \right| \leq M \left| \int_0^1 e^{n^2 x} dx \right| = \frac{M(e^{n^2} - 1)}{n^2} \quad \text{(2 marks)}$$

$$\Rightarrow 1 \leq \frac{M(1 - e^{-n^2})}{n} \to 0 \Rightarrow \#. \quad \text{(3 marks)}$$

(b) Prove that a functions $f(x)$ continuous on the interval $[a, b]$ is Riemann Integrable on $[a, b]$. [5]

Solution: Let $\varepsilon > 0$. $f(x)$ continuous on the interval $[a, b] \Rightarrow \left| f(s) - f(t) \right| < \varepsilon \ \ for \ \left| s - t \right| < \delta$ for some $\delta > 0, \ s, t \in [a, b]$.

$\Rightarrow M_i - m_i < \varepsilon, \ for \ i = 1, 2, \ldots, n$, for a a partition of $P = \{a = x_0, x_1, \ldots., x_n = b\}$ of $[a, b]$ with $\Delta x_i = x_i - x_{i-1} < \delta$, $M_i = \sup\limits_{x_{i-1} \le x \le x_i} f(x)$ and $m_i = \inf\limits_{x_{i-1} \le x \le x_i} f(x)$. (2 marks)

$\Rightarrow U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i < \varepsilon (b - a) \Rightarrow f(x)$ is Riemann Integrable on $[a, b]$.(3 marks)